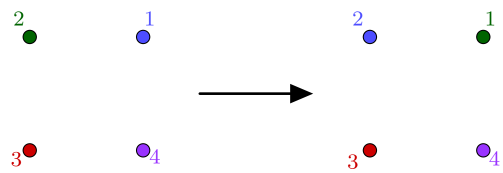
Intro to Symmetry

WDRP - Polyhedra

Consider four points labeled counterclockwise 1 to 4. Notice that we can perform an operation by moving points around like in the pictures below



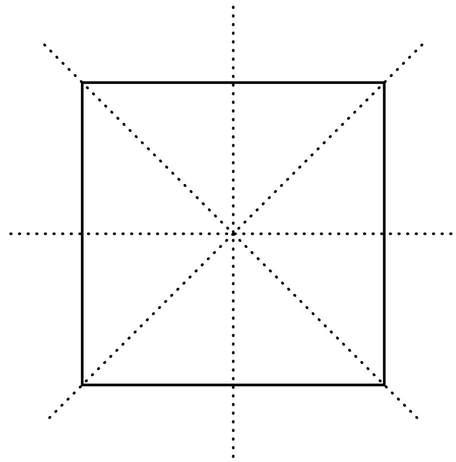
Or we could combine these two and get a new operation (Does the order we combine these operations matter?). Usually we think of symmetries as the reflection of a picture across a line being the same, think of butterflies. But this is too restrictive! The operations above are in fact symmetries, and notice from the first, which is not just a reflection, we only swap two points.

These operations are known as **permutations**. The examples only showed some permutations of four objects, but theres no reason to limit ourselves to four. For n objects we call the collection of permutations the **symmetric group on n objects** and we denote it by S_n. So in the case of our examples, they are permutations in S_4. A natural question is how many permutations are there? We could take the long route and try to come up with every permutation we can think of, which you can still do and should, as it is a good exercise. But mathematicians are lazy and would rather a simple way of coming up with an answer. So here's one way.

We can think of a permutation as first deciding where to send the point in the 1 spot. How many options do we have? (4). After we decide where 1 should go, we decide where 2 should go. How many options do we have? (3). And how many options do we have for 3? (2). The last place has to be where we send 4. You can draw a tree (yes a tree like the graph theoretic kind of tree!) of all the possibilities, try it! Each path from the root to a leaf corresponds to a permutation.

Counting them up we have 24 permutations. In general we can think of it as $4 \cdot 3 \cdot 2 \cdot 1 = 4!$ where the ! is called factorial. This can be generalized to the symmetric group for any n. Namely, the number of permutations on n objects is $n!$. We call this number the **order** of S_n.

Now, what happens if we put some lines attaching the points so we get a square. What symmetries do we have now? It's easy to see there are the four rotations, $90°, 180°, 270°, 360°$. We don't really write the last rotation as $360°$, instead we think of it as $0°$. Additionally, we have the four reflections through the dashed lines in the picture below.



This collection of symmetries is called the **dihedral group** and is denoted by D_n. It is the symmetry group for a regular n-sided polygon. In general the order of the dihedral group is $2n$, which is why some people call it D_{2n}, but we will use just n in the subscript.

If we further restrict ourselves by, say, only considering symmetries of the square which maintain the orientation, which means we get only rotations. One way we might do this restriction is by thinking of the square as having two different colors on the front and back. We could also think about this as restricting the dimension of the space that our square is in. If our square is in $3D$ space, then it has a third dimension in which it could move through to flip. But if it is restricted to $2D$ space then we have to do every symmetry by leaving it flat, which reflections do not do. The group of just rotations is called the **Cyclic group** which denoted by C_n and has order n.

To recap, we have three different groups.

1. The symmetric group S_n with order $n!$: The symmetries of n labeled points.
2. The dihedral group D_n with order $2n$: The symmetries of regular n-sided polygon.
3. The cyclic group C_n with order n: The rotations of a regular n-sided polygon.

one key property you may have already noticed is that the cyclic group is contained in the dihedral group, which is contained in the symmetric group. By extension the cyclic group is also contained in the symmetric group. We say the dihedral group and cyclic groups are **subgroups** of the symmetric group. A natural way of deriving a subgroup is by adding restrictions to how we can perform symmetries on whatever the group is the symmetries of. Like we did by adding lines, and orientation/dimension limitations. We'll see later there are other abstract ways of adding "restrictions."

In the sequel we will discuss what we mean by "symmetry" as a mathematician sees it.